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Geometry of quantum group twists, multidimensional Jackson calculus and regularization

A P Demichev†

Centro Brasileiro de Pesquisas Fisicas - CBPF/CNPq, Rua Dr. Xavier Sigaud, 150, 22290-180, Rio de Janeiro, RJ, Brasil

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Abstract. We show that R-matrices of all simple quantum groups have properties which permit one to present quantum group twists as transitions to other coordinate frames on quantum spaces. This implies physical equivalence of field theories invariant with respect to q-groups (considered as q-deformed spacetime groups of transformations) connected with each other by twists. Taking into account this freedom, we study quantum spaces of the special type, with commuting coordinates but with q-deformed differential calculus, and construct $GL_r(N)$ invariant multidimensional Jackson derivatives. We consider a particle and field theory on a two-dimensional q-space of this kind and come to the conclusion that only one (timelike) coordinate is proved to be discretized.

1. Introduction

Lattice regularization has many advantages and plays an important role in quantum field theory (see e.g. [1]). Unfortunately, it also has some shortcomings. Perhaps the most essential one is spacetime symmetry breaking. The general reason for the latter is connected with the introduction of a lattice in the theory *by hands* and therefore with its too rigid nature. Among other reasons, this fact has initiated many attempts to construct discrete ('quantized') spacetime manifolds on a deeper background (for previous attempts see e.g. [2], ch VII and references therein). In recent years this problem has made a revival and received considerable interest [3–5] due to the appearance of quantum groups (see e.g. [6] and references therein).

Quantum spaces which appear in the frame of quantum group theory [7] have many unusual properties, in particular, q-deformed differential calculi [8] and, in general, non-commuting coordinates. In one-dimensional space, a q-derivative can be represented by a Jackson difference operator [9, 10]. This, in turn, provides a description of a quantum mechanical particle on a one-dimensional lattice [11]. Thus a q-deformation of differential calculus apparently leads to space discretization. The relation of the non-commutativity of coordinates to space discretization is not so straightforward and causes problems in the construction of field theories on q-spaces. Indeed, it means that operators of coordinates cannot be diagonalized simultaneously and do not have common eigenvalues. On the other hand, asymptotic (free) states of a particle scattering process are well described by the usual non-deformed Minkowski geometry and Poincaré group representations. Usually, it is assumed that the q-group nature of the spacetime reveals itself at extremely small

† On leave of absence from Nuclear Physics Institute, Moscow State University, 119899, Moscow, Russia.

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distances and high energies. Therefore there exists a problem in this approach of how to connect the the low-energy description of particles based on a commutative geometry and the description of particles at high energy (small distances), feeling the q-deformation of the spacetime.

To clarify this point, recall how the analogous situation looks in the superstring theory [12]. Low-energy particles correspond to string zero modes. If one considers their scattering, specific for string theory, heavy modes come into play as intermediate states only, essentially improving the ultraviolet behaviour of the amplitudes but giving a negligible contribution to their finite parts. Therefore the existence of superstrings does not contradict the low-energy phenomenology based on ordinary quantum field theory.

A natural preliminary step towards understanding the relation between low-energy phenomenology and physics in a q-deformed spacetime can be the reduction of a number of non-commuting coordinates, retaining q-deformed differential calculus and q-symmetry. This work is devoted to the study of such a possibility.

As is well known, quantum spaces, related to each other by twists [13, 14] of corresponding q-groups, have different commutation relations for different coordinates [15]. The key idea of our approach is to present a group twist as a kind of q-deformed transition to other frames. As was shown in [16], a q-deformed Minkowski spacetime with non-commuting coordinates, which corresponds to a pure twisted Poincaré group (i.e. to the q-group obtained from the classical one by a twist), can be constructed from a usual Minkowski space with the help of an appropriate coordinate transformation and q-generalization of 4-beins.

In the present paper we generalize this partial result to the twists of all non-trivially deformed simple groups. More precisely, we will show that known R-matrices for all simple q-groups have a property which permits one to describe the twist procedure as a transformation of q-space coordinates.

It seems natural to require that any reasonable theory must be physically equivalent in different coordinate frames, so one can choose the most suitable frame, in particular the one with the most simple commutation relations.

To construct lattice-like regularization one definitely needs multidimensional finite differential calculus. Using the above-mentioned freedom in the choice of different q-spaces we consider $GL_q(N)$-invariant q-spaces with commuting coordinates and q-deformed differential calculus, and construct a multidimensional analogue of the Jackson calculus (invariant with respect to the appropriate quantum group).

Using the explicit formulae for a two-dimensional space, we consider a quantum mechanical particle and the simplest field theory, and show that the latter is equivalent to a system on a cylinder with a time coordinate taking values on an equidistant lattice along a cylinder. Surprisingly, the second coordinate (a spacelike coordinate on a circle) proved to be continuous. This fact has its origin in the properties of involution of the corresponding quantum general linear group which lead to construction of quantum mechanical states in a mixed coordinate–momentum representation, so that 'the second discreteness' corresponds to integer numbers which label Fourier modes on the circles.

2. Geometry of quantum group twists

As is shown in [13, 14], multiparametric quantum groups can be obtained from a one-parametric q-group via so-called twists of a quasi-triangular Hopf algebra A with the help of an element $\mathcal{F} = \sum f^i \otimes f_i \in A \otimes A$, which satisfies certain relations, so that the new coproduct $\Delta^{(F)}$ and the new universal R-matrix $\mathcal{R}^{(F)}$ are connected with the initial objects

Δ and $\mathcal{R}$ through the relations

$$\Delta^{(F)} = \mathcal{F}\Delta\mathcal{F}^{-1} \qquad \mathcal{R}^{(F)} = \mathcal{F}^{-1}\mathcal{R}\mathcal{F}^{-1}.$$

Consider at first the case of the q-deformations of $GL(N)$ groups. In this case a twist of the R-matrix in the fundamental representation R is described with the help of a diagonal matrix $F = \mathrm{diag}(f_{11}, f_{12}, \ldots, f_{nn})$ with $f_{ij} f_{ji} = 1$ so that the R-matrix $R^{(F)}$ of the twisted group $GL_{r,\tilde{q}_{ij}}(N)$ has the form

$$R^{(F)} = F^{-1} R F^{-1}.$$

Here R is (in general, also multiparametric) an R-matrix of the initial group $GL_{r,q_{ij}}(N)$ and

$$\tilde{q}_{ij} = q_{ij} f_{ij}^2. \tag{1}$$

Coordinates of the initial quantum space $C_q^N[x^i]$ satisfy the commutation relations (CRs) [7, 15, 17]

$$x^i x^j = q_{ij} x^j x^i \tag{2}$$

and coordinates of the twisted space $C_q^{(F)N}[\tilde{x}^i]$ have the CRs

$$\tilde{x}^i \tilde{x}^j = \tilde{q}_{ij} \tilde{x}^j \tilde{x}^i. \tag{3}$$

Now we introduce the algebra $E_q^N[e^i, g_j]$ with the generators $\{e^i, g_i\}_{i=1}^N$ which commute with the coordinates and put

$$\tilde{x}^i = e^i x^i \qquad \text{(no summation).} \tag{4}$$

The elements e^i play the role of components of a q-deformed (diagonal) N-bein. The CRs for them follows from (1)–(4),

$$e^i e^j = f_{ij}^2 e^j e^i \tag{5}$$

and g_i are inverse elements

$$g_i e^i = 1. \tag{6}$$

The coordinates $\tilde{x}^i$ are transformed by a q-matrix $\tilde{T}$:

$$\tilde{x}'^{\,i} = \sum_{j=1}^{N} \tilde{T}^i{}_j \tilde{x}^j. \tag{7}$$

Then using (4) and (6) one obtains from (7) transformations of the coordinates x^i:

$$x'^{\,i} = \sum_{j=1}^{N} g_i \otimes \tilde{T}^i{}_j \otimes e^j \otimes x^j. \tag{8}$$

We have used in (8) a cross product sign to stress that the elements from the different sets commute with each other (the elements g_i in (8) must be considered as the inverse elements to the generators e^i of another copy of an algebra E_q^N with respect to the elements e^i entering the same formula). Relation (8) means that the coordinates x^i are transformed by the matrix T with the entries

$$T^i{}_j = g_i \otimes \tilde{T}^i{}_j \otimes e^j \qquad \text{(no summation).} \tag{9}$$

Using (6) one can express the matrix $\tilde{T}$ through T:

$$\tilde{T}^i{}_j = e^i \otimes T^i{}_j \otimes g_j \qquad \text{(no summation).} \tag{10}$$

One can check straightforwardly that x'^i defined by (8) satisfy the correct CRs

$$x'^i x'^j = q_{ij} x'^j x'^i.$$

The general reason for this is the following property of R-matrices: if a q-matrix T satisfies a TT-relation defined by the corresponding R-matrix, then the $\tilde{T}$-matrix defined by (9) or (10) satisfies the relation with twisted R-matrix $R^{(F)}$.

To prove this statement let us write the TT-relation (CRs for entries of a matrix T) in explicit form,

$$\sum_{p,s} R^{mn}_{\ \ ps} T^p_{\ u} T^s_{\ v} = \sum_{s,r} T^n_{\ s} T^m_{\ r} R^{rs}_{\ \ uv}$$

and substitute $T^i_{\ j}$ by their expressions (9) in terms of $\tilde{T}^i_{\ j}$. This gives the relation for the latter:

$$\sum_{p,s} R^{mn}_{\ \ ps} g_p g_s e^u e^v \tilde{T}^p_{\ u} \tilde{T}^s_{\ v} = \sum_{s,r} \tilde{T}^n_{\ s} \tilde{T}^m_{\ r} g_n g_m e^s e^r R^{rs}_{\ \ uv}. \tag{11}$$

Note that in this relation the elements g_i must be considered as inverse elements of the generators e^i of another copy of an algebra E^N_q and so they commute with the elements e^i entering the same relation.

The multiparametric R-matrix for $GL_{r,q_{ij}}(N)$ group has the form

$$R^{mn}_{\ \ ps} = B^{mn}_{\ \ ps} + N^{mn}_{\ \ ps} \tag{12}$$

where B is the diagonal matrix

$$B^{mn}_{\ \ ps} = \delta^m_{\ p} \delta^n_{\ s} \big(\delta^{mn} + \Theta^{nm} q^{-1}_{mn} + \Theta^{mn} q_{nm} r^{-1} \big) \tag{13}$$

with $\Theta^{mn} = 1$ if $m > n$, $\Theta^{mn} = 0$ if $m \leqslant n$, and the matrix N is the off-diagonal part of the R-matrix,

$$N^{mn}_{\ \ ps} = \delta^m_{\ s} \delta^n_{\ p} \Theta^{mn} (1 - r^{-1}). \tag{14}$$

Using this expressions one easily obtains

$$B^{mn}_{\ \ ps} g_p g_s = g_m g_n B^{mn}_{\ \ ps} = f^{-2}_{mn} g_n g_m B^{mn}_{\ \ ps} = g_n g_m B^{(F)mn}_{\ \ \ \ \ ps} \tag{15}$$

$$N^{mn}_{\ \ ps} g_p g_s = g_n g_m N^{mn}_{\ \ ps} \tag{16}$$

so that

$$R^{mn}_{\ \ ps} g_p g_s e^u e^v = g_n g_m e^u e^v R^{(F)mn}_{\ \ \ \ \ ps}$$

where $R^{(F)}$ and $B^{(F)}$ are the twisted matrices of the same form (12)–(14) but for the twisted parameters $\tilde{q}_{ij} = q_{ij} f^2_{ij}$. Analogous consideration of the right-hand side of (11) shows that this relation can be rewritten in the form

$$\sum_{p,s} R^{(F)mn}_{\ \ \ \ \ ps} \tilde{T}^p_{\ u} \tilde{T}^s_{\ v} = \sum_{s,r} \tilde{T}^n_{\ s} \tilde{T}^m_{\ r} R^{(F)rs}_{\ \ \ \ \ uv}. \tag{17}$$

Thus, twisted q-matrices can be constructed with the help of q-deformed N-beins (5), (6) and formula (9), which is a direct generalization (q-deformation) of a relation between matrices of transformations in different coordinate frames.

In the case of the q-deformation of simple groups of the series B_N, C_N and D_N there is one more structure, namely an invariant length [7]

$$L_q = \sum_{i,j} x^i C_{ij} x^j = \sum_i l_i x^{i'} x^i$$

where $i' = N+1-i$. Values of the coefficients l_i can be found in [7] and are not essential for our consideration. To preserve L_q, components of a q-bein must satisfy the additional constraints

$$e^i e^{i'} = e^{i'} e^i = 1 \tag{18}$$

$i = 1, \ldots, N/2$ for the C_N and D_N series; $i = 1, \ldots, (N+1)/2$ for the B_N series. In particular, for the series B_N

$$e^{(N+1)/2} = 1. \tag{19}$$

These constraints reduce the number of twist parameters, which, from a geometrical point of view, define CRs for the components of the q-beins, so that the number is equal to $k(k-1)/2$, where k is rank of a group. R-matrices for the B_N, C_N and D_N series have the form

$$\begin{aligned} R^{ij}_{\ \ kl} = {} & [\delta^i{}_k \delta^j{}_l (r\delta^{ij}(1-\delta^{ii'}) + (\Theta^{ji} r q_{ij}^{-1} + \Theta^{ij} q_{ji} r^{-1})(1-\delta^{ii'})) + (r - r^{-1})\delta^i{}_l \delta^j{}_k \Theta^{ij}] \\ & + \Big[\frac{1}{r} \delta^i{}_k \delta^j{}_l \delta^{ji'} (1-\delta^{ii'}) - \Theta^{ij}(r - r^{-1}) r^{(\rho_i - \rho_j)} \epsilon_i \epsilon_j \delta^{ij'} \delta_{kl'} \\ & + \delta^i{}_{(N+1)/2} \delta^j{}_{(N+1)/2} \delta^{(N+1)/2}_k \delta^{(N+1)/2}_l \Big] \end{aligned}$$

(the last term exists for the B_N series only). Here ρ_i and ϵ_i are integer or half-integer numbers [7, 18]. Using this explicit form one can easily show that the R-matrices have the property analogous to that of the A_N groups.

Indeed, the terms in the first square brackets have a structure similar to that of the R-matrix for the $GL_{r,q_{ij}}(N)$ groups. So, literally repeating the proof for $GL_{r,q_{ij}}(N)$, we find that they are transformed properly when the elements e_i, g_j move through them (cf (15) and (16)). The terms in the second square brackets are not changed because of Kronecker symbols $\delta^{ji'}$, $\delta^{ij'}$ or $\delta^{(N+1)/2}_k$, $\delta^i{}_{(N+1)/2}$ and relations (18) and (19). They do not contain twist parameters q_{ij} and are the same for any twisted R-matrices.

Thus again the matrices $\tilde{T}$ defined by (9) and (10) satisfy the CRs (17) for twisted quantum groups.

The interpretation of twists as transitions to other q-coordinate frames is extended to differential calculi on q-spaces. Indeed, using the CRs which define a q-deformed differential calculus in the multiparametric case [15], one can straightforwardly check that the relations

$$\mathrm{d}\tilde{x}^i = e^i \, \mathrm{d}x^i \tag{20}$$

$$\tilde{\partial}_i = g_i \partial_i \tag{21}$$

convert differential calculus on a q-space to the one on a twisted q-space. For example, $GL_{r,q_{ij}}(N)$ invariant relations for coordinates and derivatives read

$$\begin{aligned} \tilde{\partial}_i \tilde{x}^i &= 1 + r\tilde{x}^i \tilde{\partial} + (r-1) \sum_{a=i+1}^{n} \tilde{x}^a \tilde{\partial}_a \\ \tilde{\partial}_i \tilde{x}^k &= \frac{r}{\tilde{q}_{ik}} \tilde{x}^k \tilde{\partial}_i \qquad i < k \\ \tilde{\partial}_k \tilde{x}^i &= \tilde{q}_{ik} \tilde{x}^i \tilde{\partial}_k \qquad i < k. \end{aligned}$$

The first set of relations is not changed under the transformations (20) and (21) due to (6), and this corresponds to the fact that these relations do not contain twist parameters q_{ij}.

Substituting (20) and (21) into the two other sets of relations one has

$$g_i e^k \partial_i x^k = e^k g_i \frac{r}{\tilde{q}_{ik}} x^k \partial_i = g_i e^k \frac{f_{ik}^2 r}{\tilde{q}_{ik}} x^k \partial_i = g_i e^k \frac{r}{q_{ik}} x^k \partial_i$$
$$g_k e^i \partial_k x^i = e^i g_k \tilde{q}_{ik} x^i \partial_k = g_k e^i f_{ik}^{-2} \tilde{q}_{ik} x^i \partial_k = g_k e^i q_{ik} x^i \partial_k \qquad i < k$$

so that ∂_i and x^k satisfy the twisted CRs

$$\partial_i x^k = \frac{r}{q_{ik}} x^k \partial_i \qquad \partial_k x^i = q_{ik} x^i \partial_k \qquad i < k.$$

The relations for differentials can be checked quite analogously.

3. q-spaces with commuting coordinates and multidimensional Jackson differential calculus

It is well known that in the one-dimensional case q-deformed differential calculus can be realized with the help of a finite difference operation called the Jackson derivative [9, 10], which has the form

$$\mathrm{D}_r f(x) = \frac{f(x) - f(rx)}{(1-r)x} \tag{22}$$

or

$$\tilde{\mathrm{D}}_r f(x) = \frac{f(r^{-1}x) - f(x)}{(r^{-1}-1)x} \tag{23}$$

with the CRs

$$\mathrm{D}_r x - rx\mathrm{D}_r = 1 \qquad \tilde{\mathrm{D}}_r x - r^{-1}x\tilde{\mathrm{D}}_r = 1$$

(throughout this paper we will assume that $r \leqslant 1$). In particular, these derivatives are suitable for the description of a quantum mechanical particle on a one-dimensional lattice [11]. To consider quantum mechanics on higher-dimensional lattices one needs a multidimensional generalization of the Jackson calculus. Such a calculus, invariant with respect to the q-group $GL_r(N) := GL_{r,1}(N)$, can be constructed in the space $C_r^N[x^i]$ with commuting coordinates x^i.

As is shown in the previous section, commuting coordinates differ from non-commuting ones by non-commuting factors e^i. The situation reminds one of a transition from the usual three-dimensional Euclidean coordinates to well known quaternions with the basis $\{\sigma_i\}_{i=1}^3$: sometimes it is convenient to include the correspondence with the coordinates $\{x^i\}_{i=1}^3$ non-commutative quaternions $\hat{x}^i := x^i\sigma_i$ (no summation). Although this transformation brings new algebraic structure and permits one to express three-dimensional rotations in a pure algebraic way, the underlying geometrical and physical structure remains the same.

This analogy leads to the conclusion that one can choose a most convenient quantum space among a set of twisted q-spaces. In particular, in the case of spaces $C_{r,q_{ij}}^N[x^i]$, the most simple choice is the spaces $C_r^N[x^i]$ with commuting coordinates. The CRs for coordinates and derivatives on this space are the following [15]:

$$x^i x^j = x^j x^i \quad \forall\, i, j \qquad \partial_i x^i = 1 + rx^i\partial_i + (r-1)\sum_{l=i+1}^{N} x^l \partial_l$$
$$\partial_i \partial_k = \frac{1}{r}\partial_k\partial_i \qquad \partial_i x^k = rx^k\partial_i \qquad \partial_k x^i = x^i\partial_k \qquad i<k, \quad i,k = 1,\ldots,N. \tag{24}$$

To develop Jackson calculus define the operators of finite dilatations

$$A_i = 1 + (r-1)\sum_{j=i}^{N} x^j \partial_j \tag{25}$$

with the commutation relations

$$\begin{aligned} &A_i A_k = A_k A_i \qquad \forall\, k, i \\ &A_k x^i = x^i A_k \qquad k > i \\ &A_i x^k = r x^k A_i \qquad k \geqslant i. \end{aligned} \tag{26}$$

Note that the operators A_i are analogous to the operators $Y^i{}_j$ of vector fields on a simple quantum group, introduced in [19]. Relations (25) permit one to express the q-derivatives in terms of A_i

$$\partial_i = (1-r)^{-1}(x^i)^{-1}(A_{i+1} - A_i) \qquad i = 1, \ldots, N, \quad A_{N+1} := 1. \tag{27}$$

Relations (26) and (27) lead to the following realization of the q-derivatives in a space of functions of N commuting variables:

$$\partial_i f(x^1, \ldots, x^N) = \frac{f(x^1, \ldots, x^i, rx^{i+1}, \ldots, rx^N) - f(x^1, \ldots, x^{i-1}, rx^i, \ldots, rx^N)}{(1-r)x^i}. \tag{28}$$

One can easily check that the finite differences (28) indeed satisfy the $GL_r(N)$ invariant CRs (24). These differences look like a natural multidimensional generalization of the Jackson derivative (22).

As is shown in [8], there are two types of CRs for q-derivatives and coordinates, which are invariant with respect to q-deformed groups. The first possibility is presented in (24), the second one is the following:

$$\begin{aligned} &\hat\partial_i x^i = 1 + \frac{1}{r} x^i \hat\partial_i + \left(\frac{1}{r} - 1\right) \sum_{l=1}^{i-1} x^l \hat\partial_l \\ &\hat\partial_i \hat\partial_k = \frac{1}{r} \hat\partial_k \hat\partial_i \qquad \hat\partial_i x^k = x^k \hat\partial_i \qquad \hat\partial_k x^i = \frac{1}{r} x^i \hat\partial_k \qquad i < k, \quad i, k = 1, \ldots, N. \end{aligned} \tag{29}$$

These relations can also be obtained from the general multiparametric ones by the transition (4) and (21) to new q-coordinates and derivatives. In this case it is natural to introduce the operators

$$\hat A_i = 1 + \left(\frac{1}{r} - 1\right) \sum_{j=1}^{i} x^j \hat\partial_j \tag{30}$$

which commute with each other and have the following CRs with the coordinates

$$\begin{aligned} &\hat A_k x^i = r^{-1} x^i \hat A_k \qquad k > i \\ &\hat A_i x^k = x^k \hat A_i \qquad k \geqslant i. \end{aligned} \tag{31}$$

These relations permit one to construct the realization of the q-derivatives $\hat\partial_i$ in terms of the finite differences

$$\begin{aligned} &\hat\partial_i f(x^1, \ldots, x^N) \\ &\quad = [f(r^{-1}x^1, \ldots, r^{-1}x^i, x^{i+1}, \ldots, x^N) - f(r^{-1}x^1, \ldots, r^{-1}x^{i-1}, rx^i, \ldots, rx^N)] \\ &\quad \times [(r^{-1} - 1)x^i]^{-1} \end{aligned} \tag{32}$$

which is the generalization of the Jackson derivative of the second type (23).

As follows from (25), (30) and (6), the A-operators are invariant with respect to transformations (4) and (21) and, hence, the A-operators have the same form and properties in all q-coordinate frames related by twists.

4. A quantum particle in two-dimensional quantum space

In this section we apply the above-considered formulae to construction of quantum mechanics on a two-dimensional quantum plane. For convenience we rewrite the CRs (24), (26), (29) and (31) in this particular case denoting $x^1 = z,\ x^2 = \bar{z},\ \partial_1 = \partial,\ \partial_2 = \bar{\partial},\ \hat{\partial}_1 = \hat{\partial},\ \hat{\partial}_2 = \hat{\bar{\partial}},\ A_1 = A,\ A_2 = \bar{A},\ \hat{A}_1 = \hat{A},\ A_2 = \hat{\bar{A}}$,

$$z\bar{z} = \bar{z}z \qquad \partial\bar{\partial} = \frac{1}{r}\bar{\partial}\partial \qquad \hat{\partial}\hat{\bar{\partial}} = \frac{1}{r}\hat{\bar{\partial}}\hat{\partial} \tag{33}$$

$$\partial z = 1 + rz\partial + (r-1)\bar{z}\bar{\partial} \qquad \bar{\partial}\bar{z} = 1 + r\bar{z}\bar{\partial} \qquad \partial\bar{z} = r\bar{z}\partial \qquad \bar{\partial}z = z\bar{\partial} \tag{34}$$

$$Az = rzA \qquad A\bar{z} = r\bar{z}A \qquad \bar{A}z = z\bar{A} \qquad \bar{A}\bar{z} = r\bar{z}\bar{A} \tag{35}$$

$$\hat{\partial}z = 1 + \frac{1}{r}z\hat{\partial} \qquad \hat{\bar{\partial}}\bar{z} = 1 + \left(\frac{1}{r} - 1\right)z\hat{\partial} + \frac{1}{r}\bar{z}\hat{\bar{\partial}} \qquad \hat{\partial}\bar{z} = \bar{z}\hat{\partial} \qquad \hat{\bar{\partial}}z = \frac{1}{r}z\hat{\bar{\partial}} \tag{36}$$

$$\hat{A}z = \frac{1}{r}z\hat{A} \qquad \hat{A}\bar{z} = \bar{z}\hat{A} \qquad \hat{\bar{A}}z = \frac{1}{r}z\hat{\bar{A}} \qquad \hat{\bar{A}}\bar{z} = \frac{1}{r}\bar{z}\hat{\bar{A}}. \tag{37}$$

All A-operators commute with each other and satisfy the relations

$$\hat{\bar{A}}A = 1 \qquad \hat{\bar{A}}\bar{A} = \hat{A} \qquad \hat{A}A = \bar{A}. \tag{38}$$

The simplest way to find them is to derive them from an action of the operators on an arbitrary function $f(z, \bar{z})$. The CRs between different types of q-derivatives have the form

$$\hat{\partial}\partial = r\partial\hat{\partial} \qquad \hat{\bar{\partial}}\bar{\partial} = r\bar{\partial}\hat{\bar{\partial}} \qquad \hat{\partial}\bar{\partial} = \bar{\partial}\hat{\partial} \qquad \hat{\bar{\partial}}\partial = r^2\bar{\partial}\hat{\bar{\partial}}. \tag{39}$$

Now we must define $*$-involution in the algebra of the operators which enter the relations (33)–(39).

We want to consider the parameter r as a lattice spacing, hence, it must be a real number. The appropriate involution for $GL_r(2)$ in this case is the following [18]:

$$T^* = CTC \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This means that

$$T^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

and $z^* = \bar{z}$. It is not difficult to see that the CRs for the A-operators and the coordinates are consistent with the involution

$$A^* = \hat{\bar{A}} \qquad \bar{A}^* = \hat{A}. \tag{40}$$

This gives the involution rules for the derivatives:

$$\partial^* = -\hat{\bar{\partial}} + \left(\frac{1}{r} - 1\right)\frac{z}{\bar{z}}\hat{\partial} + \frac{1}{\bar{z}} \qquad \bar{\partial}^* = -\hat{\partial} + \frac{1}{z}.$$

These rules look rather cumbersome, but they are a direct generalization of the involution in the one-dimensional case [11].

The next step is to construct the representation of the operators in a Hilbert space so that the involution would coincide with Hermitian conjugation. To construct a convenient basis of a Hilbert space one needs Hermitian operators. Combinations of the coordinates of the form

$$x = (z + \bar{z})/2 \qquad y = (z - \bar{z})/2\mathrm{i}$$

are not convenient as they have identical CRs with some of the A-operators and rather cumbersome CRs with others. A-operators play the role of conjugate momenta on a lattice (cf [11]). A natural choice of position operators follows from the observation that the A-operators generate finite dilatations rather than translations. This implies the introduction of the operators

$$\rho = \sqrt{\bar{z}z} \qquad \rho^* = \rho \tag{41}$$

and

$$\Phi = \sqrt{\bar{z}z^{-1}} \qquad \Phi^*\Phi = 1. \tag{42}$$

As follows from (40) and (38), we also have the operators A and $B := \hat{A}\bar{A}$ such that

$$A^*A = 1 \qquad B^* = B. \tag{43}$$

The operators have the following CRs:

$$A\rho = r\rho A \qquad B\Phi = r\Phi B \qquad A\Phi = \Phi A \qquad B\rho = \rho B. \tag{44}$$

All other operators can be expressed in terms of these four relations.

The CRs (44) play the role of canonical commutation relations (the q-analogue of two-dimensional Heisenberg algebra). As is seen from (44) we have two mutually commuting subalgebras generated by the pairs of the operators A, ρ and B, Φ (the analogue of canonically conjugate momenta and coordinates). Their matrix representations are constructed on a common domain D_{ρ_0,b_0} consisting of all linear combinations of vectors $|N,m\rangle_{\rho_0,b_0}$:

$$\rho|N,m\rangle_{\rho_0,b_0} = \rho_0 r^{-N}|N,m\rangle_{\rho_0,b_0} \qquad B|N,m\rangle_{\rho_0,b_0} = b_0 r^{-m}|N,m\rangle_{\rho_0,b_0} \tag{45}$$

$$A|N,m\rangle_{\rho_0,b_0} = |N+1,m\rangle_{\rho_0,b_0} \qquad \Phi|N,m\rangle_{\rho_0,b_0} = |N,m-1\rangle_{\rho_0,b_0}. \tag{46}$$

The constants ρ_0 and b_0 mark different representations and, from the eigenvalues of ρ and B, it follows that in the ranges $[\rho_0, r\rho_0)$ and $[b_0, rb_0)$ the representations are inequivalent. The matrices ρ, B are Hermitian and A, Φ are unitary with respect to the scalar product defined by

$${}_{\rho_0,b_0}\langle N,m|N',m'\rangle_{\rho_0,b_0} = \delta_{NN'}\delta_{mm'}.$$

As usual, D_{ρ_0,b_0} can be completed to a Hilbert space $\mathcal{H}_{\rho_0,b_0}$

$$\mathcal{H}_{\rho_0,b_0} = \left\{ \sum_{N,m\in\mathbb{Z}} C_{Nm}|N,m\rangle_{\rho_0,b_0} : \sum_{N,m\in\mathbb{Z}} |C_{Nm}|^2 < \infty \right\}.$$

The operators ρ and B are essentially self-adjoint in the Hilbert space; their self-adjoint extension is defined on the domain

$$D^+_{\rho_0,b_0} = \left\{ \sum_{N,m\in\mathbb{Z}} C_{Nm}|N,m\rangle_{\rho_0,b_0} : \sum_{N,m\in\mathbb{Z}} |C_{Nm}|^2 r^{-2N} < \infty, \quad \sum_{N,m\in\mathbb{Z}} |C_{Nm}|^2 r^{-2m} < \infty \right\}.$$

The operators A, ϕ can be easily extended to unitary operators with the domain $\mathcal{H}_{\rho_0,b_0}$ (see e.g. [4]; notice that the algebra considered in this work contains the subalgebra generated by the operators denoted there by p and u which is isomorphic to the algebra of A, ρ or B, Φ and has the same involution properties). Therefore in what follows we will denote the involution by a sign of Hermitian conjugation.

We will consider one of the representations labelled by ρ_0, b_0 and for shortness put $\rho_0 = b_0 = 1$ (one can always achieve these values by appropriate rescaling of the operators $\rho \to \rho_0\rho$ and $B \to b_0 B$, the defining CRs (44) being invariant with respect to this transformation).

So, starting from the $GL_r(2)$ invariant differential calculus on a quantum plane, we have come naturally to polar coordinates, the operator ρ being the operator of radial coordinates and the operator Φ being of the form $\Phi = \mathrm{e}^{-\mathrm{i}\phi}$, where ϕ is an (Hermitian) operator of an angle coordinate. The structure of the algebra involution leads to the mixed representation with one coordinate (ρ) and one momentum diagonal operator. In the classical case a multiplication of a function by Φ shifts its Fourier component numbers by minus unity. This corresponds to the action (46) of the operator Φ on vectors of $\mathcal{H}$. This implies that B is connected with an angular momentum operator. In fact it is the analogue of the operator of the form

$$\exp\left\{\mathrm{i}\phi_0\frac{\partial}{\partial\phi}\right\} = \exp\{\phi_0 M\} \tag{47}$$

in the case of quantum mechanics on the usual continuous plane, where ϕ_0 is some fixed angle and $\mathrm{i}\partial/\partial\phi$ is the (two-dimensional) angular momentum operator. Eigenfunctions of this operator are periodic functions $\exp\{\mathrm{i}n\phi\}$ with eigenvalues $\exp\{n\phi_0\} = (\mathrm{e}^{\phi_0})^n$. The latter expression coincides with the eigenvalue of operator B if one equates $r = \exp\{\phi_0\}$.

Thus the operator ρ defines values of the radial coordinate and the A operator shifts them (it plays the role of conjugate momentum). Analogously, the operator B defines angular momentum values and the operator Φ of the conjugate coordinate shifts its eigenvalues.

Now we can consider a q-subgroup Λ of the $GL_r(2,\mathbb{R})$ of matrices of the form

$$T = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix}$$

with $aa^* = a^*a$. Because of the latter relation it looks like an ordinary group isomorphic to a multiplicative group of complex numbers, but one must remember about the CRs with the generators $Y^i{}_j$ of the corresponding quantum universal enveloping algebra. For left-invariant and right-covariant generators they have the following general form [19],

$$Y^i{}_j T^k{}_s = T^k_l Y^m_n (\hat{R}_{21})^{il}{}_{mt} (\hat{R}_{12})^{nt}{}_{js} \tag{48}$$

where $\hat{R}_{12}$ and $\hat{R}_{21}$ are properly normalized R-matrices of the $GL_r(2,\mathbb{R})$. The CRs (48) give for the subgroup Λ (i.e. if $b = c = 0$)

$$Y^1{}_1 a = raY^1{}_1 \qquad Y^1{}_1 a^* = a^*Y^1{}_1 \qquad Y^2{}_2 a = aY^2{}_2 \qquad Y^2{}_2 a^* = ra^*Y^2{}_2. \tag{49}$$

It is easy to see that the CRs for $\mathcal{D} := Y^1{}_1 Y^2{}_2$, $\bar{\mathcal{D}} := Y^2{}_2$, a and a^* are the same as for $A, \bar{A}, z, \bar{z}$. So the algebra (35) on the quantum plane is isomorphic to that of the q-subgroup Λ and we can identify the q-plane with this subgroup. The operators $\hat{A}$ and $\hat{\bar{A}}$ correspond to right-invariant and left-covariant generators (cf [19]). On the other hand, the q-subgroup Λ can play the role of symmetry group, the whole $GL_r(2,\mathbb{R})$ group being the group of linear canonical transformations (of CRs (34) and (36)).

The coaction of Λ on the coordinates is

$$\begin{aligned} &z \to z' = a\otimes z \qquad \bar{z} \to \bar{z}' = a^*\otimes\bar{z} \\ &\rho \to \rho' = a_\rho \otimes \rho \qquad \Phi \to \Phi' = a_\Phi \otimes \Phi \\ &a_\rho = \sqrt{a^*a} \qquad a_\Phi = \sqrt{a/a^*}. \end{aligned} \tag{50}$$

The coordinates ρ' and Φ' have a representation in a Hilbert space $\mathcal{H}\otimes\mathcal{H}$ (cross product of the same Hilbert space, as in our case the comodule coincides with the symmetry q-group) which has a subspace (diagonal) isomorphic to $\mathcal{H}$. The latter corresponds to the representation of the CRs based on the coordinates ρ' and Φ' as primary ones. All A-operators, and therefore the operator B, are invariant with respect to the coaction (50).

To consider a two-dimensional quantum free particle one can use, for example, the Hamiltonian

$$h = \frac{\Omega}{(1-r)^2}[(1-A)(1-A^\dagger) + (1-B)^2] \tag{51}$$

where Ω is constant with dimension of energy. In order to find eigenfunctions and eigenvalues of the Hamiltonian introduce the operators

$$\rho^{\mathrm{i}P} = \exp\{\mathrm{i}P \ln \rho\}$$

and the states (cf [20])

$$|\mathbb{1}, m\rangle = \sum_{m=-\infty}^{\infty} |N, m\rangle$$

$$|P, m\rangle = \rho^{\mathrm{i}P} |\mathbb{1}, m\rangle$$

with the properties

$$A|\mathbb{1}, m\rangle = |\mathbb{1}, m\rangle$$

$$A|P, m\rangle = r^{\mathrm{i}P}|P, m\rangle$$

where P is a real number: $0 \leqslant P \leqslant \pi/\chi$, $\chi := \ln r$. For eigenvalues of the Hamiltonian (51) we obtain

$$h|P, m\rangle = \xi_{P,m}|P, m\rangle$$

$$\xi_{P,m} = \Omega\left(\frac{(1-\cos\chi P)}{(1-r)^2} + [m;r]^2\right) \underset{r\to 1}{\longrightarrow} \Omega(P^2 + m^2) \qquad [m;r] = \frac{1-r^m}{1-r}.$$

Thus the operator h has the correct continuous limit but its eigenvalues are not invariant with respect to the reflection $m \to -m$. This means that the left and right modes have different properties and positive modes have a decreasing spectrum which can lead to additional divergencies in the corresponding field theories. These properties are caused by the exponent-like form of the operator B analogous to (47) as we discussed above. It is therefore natural to consider the Hamiltonian

$$H = \frac{\Omega}{(1-r)^2}[(1-A)(1-A^\dagger) + \ln^2 B]$$

with the eigenvalues $\Omega\lambda_{P,m}$, where

$$\lambda_{P,m} = \left(\frac{(1-\cos\chi P)}{(1-r)^2} + \frac{\chi^2}{(1-r)^2}m^2\right) \underset{r\to 1}{\longrightarrow} (P^2 + m^2). \tag{52}$$

As we noted at the end of the section 3, A-operators have the same properties in all coordinate frames related by twists. So properties, in particular the spectrum, of the Hamiltonian H, constructed by these operators, do not depend on the specific choice of the coordinates.

To construct two-dimensional quantum field theory we need a kind of integral over the variable ρ. A one-dimensional q-integral has been constructed in [11, 20] and we can use these results to define

$$\begin{aligned}\int_0^K \mathrm{d}_r\rho\, f(\rho) &:= \langle K, m|\partial_\rho^{-1} f(\rho)|\mathbb{1}, m\rangle \\ &= (1-r)\langle K, m|(1-A)^{-1}\rho f(\rho)|\mathbb{1}, m\rangle \\ &= (1-r)\sum_{l=-\infty}^{K} r^{-l} f(r^{-l})\end{aligned} \tag{53}$$

where the derivative ∂_ρ is defined as in (27):

$$\partial_\rho := \frac{1}{(1-r)\rho}(1-A).$$

The last expression for the integral in (53) has the form of a usual Jackson integral [10].

To construct the action, note that the two-dimensional space with the coordinates z and $\bar{z}$ and the symmetry transformations (50) can be considered as a result of the conformal map

$$u \to z = \mathrm{e}^u \qquad \bar{u} \to \bar{z} = \mathrm{e}^{\bar{u}} \tag{54}$$

of a cylinder with coordinates $\bar{u}$ and u. This map is well known and is used widely, in particular, in conformal and string theories (in the frame of so-called radial quantization, see e.g. [12]). A coordinate along the cylinder is associated with a time coordinate and a space coordinate takes values on a circle. After the conformal mapping the coordinate ρ plays the role of time and Φ that of space coordinate. It is easy to see that in coordinates u and $\bar{u}$ the transformations (50) become translations and the time coordinate $\tau := \mathrm{Re}\, u$ takes values on an equidistant lattice with a spacing $\ln r$. In the continuous case, a field theory free action for a scalar field Ψ in polar coordinates on the z-plane has the form

$$\begin{aligned} S_0 &= \int_0^\infty \int_0^{2\pi} \frac{\mathrm{d}\rho}{\rho} \mathrm{d}\phi\, \Psi(\rho,\phi)(\rho\partial_\rho\rho\partial_\rho + \partial_\Phi^2)\Psi(\rho,\phi) \\ &= \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{\mathrm{d}\rho}{\rho}\, \Psi_{-m}(\rho)(\rho\partial_\rho\rho\partial_\rho - m^2)\Psi_m(\rho) \end{aligned} \tag{55}$$

where $\Psi_m(\rho)$ are Fourier components of the field $\Psi(\rho,\phi)$. This expression is explicitly invariant with respect to dilatations of ρ and translations of ϕ (the continuous analogue of (50)).

A quantum plane analogue has the following form:

$$S_0^r = \lim_{K\to\infty} \sum_{m=-\infty}^{\infty} \langle K, m|(1-A)^{-1}\Psi_{-m}(\rho)G^{-1}\Psi_m(\rho)|\mathbb{1}, m\rangle.$$

Here the operator G^{-1} has a form similar to the Hamiltonian H

$$G^{-1} = \frac{H}{\Omega}$$

though its meaning is quite different, of course. The finite difference operator G^{-1} in this action has the eigenvalues $\lambda_{P,m}$ presented in (52). In the case of interacting Ψ^4 theory, a one-loop correction to, for example, the mass term is proportional to a trace of the operator G,

$$\sum_{m=-\infty}^{\infty} \int_0^{-\pi/\ln r} \frac{\mathrm{d}P}{\lambda_{P,m}}$$

which has no divergencies. Note, however, that the summation over orbital number m is the same as in the continuous case.

5. Conclusion

The main result of this work is the statement that twists of quantum groups and corresponding q-spaces can be realized with the help of the auxiliary non-commuting elements e^i, g_k satisfying relations (5) and (6). Geometrically this can be interpreted as a transition to another coordinate frame on a q-space and so for many problems twisted quantum groups must be physically equivalent. Let us mention again the analogy between the algebra of e^i, g_k and well known quaternions σ_i: one can use either usual three-dimensional coordinates x^i or non-commutative quaternions $\hat{x}^i := x^i\sigma_i$ (no summation). The physical content of a problem is not changed. Unfortunately, many important quantum groups, for example orthogonal ones, do not have twisted counterparts with commuting coordinates of a corresponding quantum space, but even in these cases the freedom can be used for the choice of a most convenient q-coordinate frame. Note also that linear groups play an important role as groups of spacetime symmetries. It is enough to mention $SL(2, C)$ as a universal covering of the Lorentz group and $SU(2, 2)$ as a covering of the four-dimensional conformal group.

Using q-deformed space with commuting coordinates we have constructed a multidimensional q-group invariant generalization of the famous Jackson derivatives.

As another application of q-spaces with commuting coordinates we have considered quantum mechanics and simple field theory on a two-dimensional quantum space. The structure of the involution leads to the mixed coordinate–angular momentum representation of states of the system. This, in turn, results in discretization of only one (radial) coordinate of the space in spite of the q-deformed differential calculus, the spectrum of the angular momentum operator being unbounded as in the usual continuous case. Such partial discretization of a spacetime is enough for the regularization of two-dimensional models but in higher dimensions such field theory can by ultraviolet divergent even in a q-spacetime. Another lesson learned from the considered models is that q-derivatives or finite differences, constructed with the help of operators of the form (25), are connected with dilatations and not with translations. This implies that the corresponding coordinates are related with the usual ones (in which derivatives generate translations) by a nonlinear exponent-like map of the type (54). It is clear that symmetry transformations, for example of the Lorentz group, have quite a different form in nonlinearly transformed coordinate frames. This remark might be important for numerous attempts to construct quantum deformations of relativistic symmetry.

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